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HYPERVELOCITY IMPACT OF MICROPARTICLES

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CONSIDERATIONS ON ERRONEOUS FLOW EQUATIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

INTRODUCTION

This is the nineteenth Quarterly Progress Report on Project NASr-7. The title of this project is "Shock, Flow and Radiation from the Hypervelocity Impact of Microparticles". The analytical and experimental phases of this project are concerned with the phenomena that occur with the hypervelocity impact of microparticles on a massive target, which may be solid or layered. The overall objective is to obtain a sufficiently fundamental understanding of the physical mechanisms in the hypervelocity impact of a microparticle so that the momentum and the energy transferred from the particle to the target may be ascertained, and possibly, the density of the incident microparticles may be estimated.

The two incorrect Progress Reports, Nos. 17 and 18, were forwarded with comments that ignore the basic errors in them. This was considered acceptable provided the distribution was restricted and the errors were discussed in the next report. It is most embarrassing to have these incorrect reports floating around without comments that correctly evaluate them. As it develops, some dependence may be placed on the curves in these reports. With some rewriting of the reports, the curves could be used to illustrate the problem of an infinitely long cylinder which is incident at hypervelocity on a semi-infinite slab. The criticisms of the thesis which is Progress Reports Nos. 17 and 18 is equally valid as a criticism of the thesis which was forwarded as Progress Report No. 13.

The Progress Reports that are mentioned above are in error when reference is made to the impact of a sphere. The reports are almost, but not quite correct, provided the nomenclature is changed to claim a solution for the hypervelocity impact of an infinitely long, aluminum cylinder on a semi-infinite slab of aluminum. The qualitative features of the impact are correct but the quantitative values are in error. This is attributable to the omission of one term from each of the two momentum equations. One momentum equation is for the radial component of the momentum and the other one is for the transverse component of the momentum.

In the following report, the errors in Progress Reports 17 and 18 are always cited. The equations in Progress Report 13 have the same basic errors and some additional errors from carelessness with the transcription, but no cite them would be confusing. The following report starts with a consideration of the errors in the flow equations for the conservation of mass, momentum and energy. In each case, the "conservative" equations in the report are converted to the component form of the flow equations that are given by Rae[3]. A cross-check is established when the vector form of the equations of flow by R. D. Richtmyer are converted to Rae's form of the equations. These considerations are confined to cylindrical coordinates.

After consideration of each of the conservation equations, the component forms of the equations are essembled in one section for future reference. The equations are given in spherical coordinates but are not derived. The reduction from the "conservative" to the component form belongs in the revised thesis. The initial boundary conditions are considered and it is shown that those in the thesis are for an infinitely long cylinder. The "conservative" form of the equations for computer calculations are discussed and speculations are made on the intrinsic value of the solutions that were forwarded in Progress Reports 13, 17 and 18. The evaluation must be designated as speculative until the actual solutions are obtained because of the available information. In this connection, it may be mentioned that Bjork et al[7] employed the calculated results from the oblique, hypervelocity impact of an infinitely long cylinder to estimate the effect from the impact of a sphere.

DEFINITION OF UNITS OF ENERGY

The criticisms and the corrections are given with page references to the unaccepted Ph.D. thesis which is Progress Report No. 17. The first equation for reference is equation 2.3 on page 10

$$\rho \frac{d\varepsilon}{dt} = -\nabla\cdot(pu) \tag{1}$$

In this equation, the term ε is correctly defined as "the total energy per unit mass". The practical units are ergs per gram. This equation is correct but the statement is not the most useful for further calculations. It will be converted to a more common form below. The second equation for reference is equation 2.4 on page 10

$$p = p\ (\rho, e) \tag{2}$$

This equation is correct when e is the total internal energy per unit mass. Observe the difference in the units in these two equations. The preceding equations and all other equations up to page 10 in the report are correct. Considerable routine calculation is required to show that Equation 1 is correct. This calculation will be given below. The proof of this relation is straightforward but is very long and is placed in Appendix I.

CONSERVATION OF MASS

The first erroneous equation, as presented in the report, is equation 2.5 on page 12. This equation states

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(r\rho u)}{r\partial r} - \frac{\partial(\rho w)}{r\partial\theta} \tag{3}$$

The preceding equation is the equivalent of the vector equation for the conservation of mass in cylindrical coordinates that is given in equation a in Appendix A. The proof that the preceding equation is equivalent to the vector equation from Richtmyer and to the components equation from Rae is given in Appendix B. In terms of the component equation, Equation 3 is found in the appendix to have the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} + \rho\left[\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r}\right] = 0 \tag{4}$$

This equation is correct for cylindrical coordinates. For spherical coordinates with no flow in the direction of $r \sin\theta \, \Delta\phi$, the equation becomes

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} + \rho\left[\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{2u}{r} + \frac{w}{r}\cot\theta\right] \tag{5}$$

By a comparison of Equations 4 and 5, the equation in spherical coordinates has the added terms

$$\rho\left[\frac{u}{r} + \frac{w}{r}\cot\theta\right]$$

CONSERVATION OF MOMENTUM

The next incorrect equation is the unnumbered relation at the top of page 13 in the thesis. There are two errors in this equation. First, a term, $\frac{1}{\rho}$, should multiply the last term on the left side of the equality sign; i.e. the last term should read $\frac{1}{\rho}\frac{\partial p}{\partial r}$. The second error in this relation is the omission of a term that will be named in reference to the next equation in the thesis. Since this relation is not required in this proof and is not used again in the thesis, no further reference will be made to this relation.

The next equation in the report is the first of the two conservation of momentum equations. The two momentum equations are the radial and the tangential components for the vector form of the conservation of momentum equation. The equation for the radial component of the velocity, u, is Equation 2.6 on page 13. This equation is in error as it appears in the thesis. It is important to note that the corrected equation is valid for both cylindrical and for symmetrical flow in spherical coordinaties. By symmetrical flow in spherical coordinates is indicated a flow in which the component in the direction of $r \sin\theta \, \Delta\phi$ is zero. Equation 2.6 in the thesis is in error by the omission of a term. The incorrect equation in the text, Equation 2.6 on page 13, is

$$\frac{\partial(\rho u)}{\partial t} = - \frac{\partial p}{\partial r} - \frac{\partial(r\rho u^2)}{r\partial r} - \frac{\partial(\rho w u)}{r\partial\theta} \tag{6}$$

To be correct for cylindrical coordinates and for symmetrical flow in spherical coordinates, it should be written

$$\frac{\partial(\rho u)}{\partial t} = - \frac{\partial p}{\partial r} - \frac{\partial(r\rho u^2)}{r\partial r} - \frac{\partial(\rho w u)}{r\partial\theta} + \frac{w^2}{r} \tag{7}$$

The proof of this relation is obtained by the use of the relation in Equation 3, above. The correct equation for components is taken from a paper by William J. Rae and Henry P. Kirchner[4] that was presented at the Sixth Conference on Hypervelocity Impact. The correct equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial \theta} - \frac{w^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{8}$$

The proof of the equivalence of Equations 7 and 8 is given in the first part of Appendix C.

The second, and last component of the momentum relation is Equation 2.7 on page 13. This relation is for the w component of the velocity which is in the tangential direction, $r \Delta \theta$; i.e. in a plane that contains the origin and the cylindrical radii, r. As was true for the other component of the velocity, that is the u-component, the correct form of this relation applies for both cylindrical coordinates and for spherical coordinates provided there is no flow in the direction of $r \sin\theta \, \Delta\phi$ for the spherical coordinates. The incorrect momentum equation for the velocity component, w, is written

$$\frac{\partial(\rho w)}{\partial t} = - \frac{\partial p}{r\partial\theta} - \frac{\partial(r\rho uw)}{r\partial r} - \frac{\partial(\rho w^2)}{r\partial\theta} \tag{9}$$

To correct this equation for both cylindrical and spherical coordinates with symmetrical flow, a term must be subtracted. The correct equation is

$$\frac{\partial(\rho w)}{\partial t} = - \frac{\partial p}{r\partial\theta} - \frac{\partial(r\rho uw)}{r\partial r} - \frac{\partial(\rho w^2)}{r\partial\theta} - \frac{uw}{r} \tag{10}$$

The proof of this relation is easily obtained by the use of Equation 3, above. The correct equation is given by Rae and Kirchner[4] and is

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{r\partial\theta} + \frac{uw}{r} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta} = 0 \tag{11}$$

The proof of the equivalence of Equations 10 and 11 is given in Appendix C.

CONSERVATION OF ENERGY

There are two forms of the equation for the conservation of energy in the thesis. The first relation is Equation 2.3 on page 10 of the thesis. This equation is correct. It was first written as Equation 1 in this report and was shown to be correct in Appendix A. This equation is written

$$\rho \frac{d\epsilon}{dt} = -\nabla \cdot (p\vec{u}) \qquad (1)$$

where $\vec{u}$ is the vector velocity of flow, p is the pressure, ρ is the density and, ϵ, is the total energy. Observe that the total derivative is required on the left side of this equation. The relation between the total energy per unit mass, ϵ , and the internal energy per unit mass, e, in cylindrical coordinates with no flow in the direction, ΔZ, is

$$\epsilon = e + \frac{1}{2} u^2 + \frac{1}{2} w^2 \qquad (12)$$

where u is the component of the velocity in the direction, Δr , and w is the component of the velocity in the direction, $r\Delta\theta$. In Appendix A, it was shown that Equation 1, above, can be converted to the vector form of the equation for the conservation of energy that is given by Richtmyer[5], and which is given as Equation c in Appendix A.

As more of the text of the thesis is considered, the conservation of energy, Equation 28 on page 13, is wrong, as it is written. The error is one of carelessness in copying. In this equation, the symbol, e, is used without definition, at the equation. A definition of the symbol, e, is inherent in Equation 2 of this report, which is Equation 2.4 on page 10 of the thesis. By employing the symbol, e, in the equation of state, it must be defined as the internal energy and it may be defined as the internal energy per unit mass. The equation from the thesis is

$$\frac{\partial(\rho e)}{\partial t} = - \frac{\partial(r p u)}{r \partial r} - \frac{\partial(p w)}{r \partial \theta} - \frac{\partial(r \rho u e)}{r \partial r} - \frac{\partial(\rho w e)}{r \partial \theta}$$

This equation is wrong, as it is written. The correct equation should have ϵ instead of e; i.e. the total energy instead of the internal energy. The total energy is the sum of the internal energy and the kinetic energy as is shown by Equation 12, above. The correct form of the preceding equation is

$$\frac{\partial(\rho \epsilon)}{\partial t} = - \frac{\partial(r p u)}{r \partial r} - \frac{\partial(p w)}{r \partial \theta} - \frac{\partial(r \rho u \epsilon)}{r \partial r} - \frac{\partial(\rho w \epsilon)}{r \partial \theta} \qquad (13)$$

The proof that this equation is correct and reduces to Rae's[3] equation is presented in the first part of Appendix D. The component form of Rae's equation is

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial \theta} - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{w}{r}\frac{\partial \rho}{\partial \theta} \right) = 0 \qquad (14)$$

In the second part of Appendix D, it is shown that the vector form of the equation for the conservation of energy from Richtmyer[1] reduces to Rae's component equation[3] which is reproduced as Equation 14, above.

In the course of the preceding mathematical derivations, the extent of the validity of Equation 13 became apparent. The equation is correct for cylindrical coordinates when the flow is zero in the direction, ΔZ. There may be flow in both of the directions Δr and $r\Delta\theta$. The equation is also correct for spherical flow when there is no flow in the tangential direction, $r \sin\theta \Delta\phi$.

CORRECT FLOW EQUATIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

The preceding discussion is concerned with collecting and proving the accuracy, or error, in the equations for hydrodynamic flow in Eulerian form. With the direct approach to this problem, the discussion has scattered the correct equations over several pages. The equations are collected below so the slight deviations between the equations for cylindrical and spherical flow are easily shown. The spherical equations are not specifically derived in this report, since they are available from Rae.

A basic assumption is made that the flow of solids under the tremendous forces from a hypervelocity impact may be calculated with the equations for non-viscous, hydrodynamic flow. Two different configurations of the impacting bodies are to be considered. One problem is the hypervelocity impact of an infinitely long cylinder onto a semi-infinite slab with the axis of the cylinder parallel to the surface of the slab. The second problem is the hypervelocity impact of a sphere on a semi-infinite slab. The flow equations should be in different coordinate systems for the most direct solution of these two problems.

Eulerian Flow Equations in Cylindrical Coordinates

For the hypervelocity impact of an infinitely long cylinder, cylindrical coordinates are used but there is no flow in the direction of the axis of the cylinder, which is taken to be in the z-direction with the other coordinates as r and θ. The Eulerian equations of flow are collected from the preceding parts of this report.

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} + \rho\left[\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r}\right] = 0 \qquad (4)$$

Conservation of Momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{r\partial \theta} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{w^2}{r} = 0 \qquad (8)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{r\partial \theta} + \frac{1}{\rho}\frac{\partial p}{r\partial \theta} + \frac{uw}{r} = 0 \qquad (10)$$

Conservation of Energy

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + w\frac{\partial e}{r\partial \theta} - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta}\right) = 0 \qquad (14)$$

Two of the equations in Sodek's thesis deviate from the above equations which are for an impacting cylinder, and not for a sphere as the thesis states. The deviations are in the two equations for the conservation of momentum. The term, $-\frac{w^2}{r}$, is omitted from Equation 8 and the term $+\frac{uw}{r}$ is omitted from Equation 10. The equations for the conservation of mass and of energy are the same as in the thesis. The deviations of the thesis equations from those for an impacting sphere are discussed below.

Eulerian Flow Equations in Spherical Coordinates

For the hypervelocity impact of a sphere on a semi-infinite slab, spherical coordinates are the most practical but the flow is entirely radial so the variables are r and θ. There is no tangential flow in the direction of $r \sin\theta \, \Delta\phi$. The Eulerian equations of flow are given by Rae[3] and have not been specifically proved in this report.

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} + \rho\left[\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{2u}{r} + \frac{w}{r}\cot\theta\right] = 0 \qquad (15)$$

Conservation of Momentum, no change from cylindrical

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{r\partial \theta} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{w^2}{r} = 0 \qquad (8)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{r\partial \theta} + \frac{1}{\rho}\frac{\partial p}{r\partial \theta} + \frac{uw}{r} = 0 \qquad (10)$$

Conservation of Energy, no change from cylindrical

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + w\frac{\partial w}{r\partial \theta} - \frac{p}{\rho^2}\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta}\right] = 0 \qquad (14)$$

It is to be observed that there is a change in only the first equation from the same flow equations in cylindrical coordinates. This means, however, that three of the four equations for hydrodynamic flow in the thesis are not correct for an impacting sphere. The momentum equations have the same error as for an impacting cylinder. The equations for the conservation of mass requires the addition of two terms, $\frac{u}{r} + \frac{w}{r} \cot \theta$.

The flow equations are not, of course, the only factors that affect the solution. The initial boundary conditions for the cylindrical and spherical impact are quite different and have a major affect on the solution. The boundary conditions are presented and discussed in the next section. After these are presented, there is a short discussion of the significance of the solution that is obtained in the thesis.

INITIAL BOUNDARY CONDITIONS

In the thesis, the initial values and the initial boundary conditions are mentioned in several places. The most extensive discussion is on pages 39 to 42, inclusive. There is additional discussion and the very pertinent Figure 8 on pages 51 to 53, inclusive. There are some other indirect references in connection with the difference equations. The sketch in Figure 8 does not have the correct proportions, but it does illustrate pertinent features of the problem. The discussions in these two places in the thesis are not entirely explicit, but only a very little additional information is required from the author of the thesis in order to ascertain the situation.

The position of the origin of the coordinate system must be defined with respect to the point of impact. Refer to Figure 8 in the thesis for the position of the coordinate system. The origin is taken to be at a distance, 3R, above the plane of impact--this is not the proportions in Figure 8. The radius of the impacting sphere is defined as R. The radius of the coordinate system, r, is measured from the origin. The angle, $\theta=45^\circ$, is shown in Figure 8. Straight down in this figure corresponds to $\theta=90^\circ$.

In the computer calculations, only the range from $\theta=90^{\circ}$ to $\theta=40$ to 35° is programmed for the computer. The computer is programmed to consider meshes from $\theta=90^{\circ}$ to $\theta=45^{\circ}$. If the disturbance from the impact reaches the edge of the meshes at $\theta=45^{\circ}$, the computer extends the meshes to $\theta=40^{\circ}$, etc. For cylindrical coordinates, the z-axis would be perpendicular to the paper.

For the specified coordinate system, the surface of the semi-infinite slab is below the origin at the position

$$r \sin\theta = 3R \tag{15}$$

where R is the radius of the impacting cylinder, or sphere. It is to be observed that a little difference in the notation exists between the above equation and Equation 3.12 on page 39 in the thesis, but the difference is obvious.

The equation of the trace of the circumference of the sphere, or cylinder, on a plane through the point of contact and which contains the normal axis of the sphere, or cylinder, is given by equation 3.13 on page 40 of the thesis. The equation is

$$b^2 - a^2 + R^2 - 2bR \sin\theta = 0$$

where a is the radius of the circle which has its center at $(b,\frac{\pi}{2})$. In the nomenclature for the flow equations and for Equation 15, above, the required substitutions are b = 2R, a = R and R = r. With these new variables, the equation of the trace becomes

$$4R^2 - R^2 + r^2 - 4Rr \sin\theta = 0$$

or

$$r^2 - 4Rr \sin\theta + 3R^2 = 0 \tag{16}$$

In the thesis, there is no further direct consideration of this equation. Attention was centered on programming the boundary of the circle to move into the semi-infinite solid without disintegrating, or feathering out beyond the defining edge.

As the circle moves, the method of Rich[6] was followed to calculate the partial areas in each mesh which are occupied by matter and the part of the mesh that is empty. The thickness of all meshes was taken as the same. From these considerations and from discussions with the author of the thesis, it is known that the impacting shape was taken to be a thin, flat cylinder of constant thickness. This is the initial boundary condition for an infinitely long cylinder, and not for a sphere. The volume of each mesh, as used in the thesis, is

$$\Delta r \; r\Delta\theta \; 1 \tag{17}$$

where the distance along the z-axis is taken as unity instead of Δz.

For an impacting sphere, with the coordinates and the sphere positioned as in Figure 8 on page 52 of the thesis, the volume of each mesh is obtained by multiplying the surface area by the thickness, $r \sin\theta \; \Delta\phi$. The volume of the mesh becomes

$$\Delta r \; r\Delta\theta \; r_c \sin\theta \tag{18}$$

where $\Delta\phi$ is taken equal to unity. In the difference equations, r_c, should be measured to the "center" of the mesh. This correction is included in the recalculation of the hypervelocity impact of a sphere, which is now in progress.

To summarize these considerations on the boundary conditions that were introduced into the difference equations and the computer are those for an infinitely long cylinder, impacting at hypervelocity on a semi-infinite slab. The axis of the cylinder is parallel to the surface of the solid. The initial boundary conditions are not acceptable for the impact of a sphere. With the initial boundary conditions corresponding to an infinitely long cylinder, and with two of the four flow equations correct for cylindrical coordinates, the results in the thesis approximate more closely to the impact of an infinitely long cylinder than to the impact of a sphere. It is interesting to speculate on the magnitude of the error in the solution for the impacting cylinder.

"CONSERVATIVE" FORM OF EQUATIONS FOR CALCULATION ON A DIGITAL COMPUTER

The flow equations, in the form in which Rae[3] presents them, are not in the best form for calculations with a digital computer. The equations should be converted to the "conservative" form, speaking in computer terminology. The equations that were presented in Sodek's thesis are in the "conservative" form and the considerable exercises in vector algebra in this report are required to convert them back to the simple form that is familiar and which are given by Rae. Since the computer equations are always written, if feasible, in "conservative" form, errors and small deviations from the standard forms of these equations are only obvious after an exhaustive calculation. This is the only excuse for missing the errors in the thesis.

In the last section of this report, it was proposed to speculate on the errors that probably resulted from the use of the incorrect equations. Before proceeding with those speculations, it is desirable to clarify and emphasis the computer programmer's interest in obtaining the equations in the "conservative" form. The clarification proceeds rapidly and with little effort by reference to the specific example that follows.

Consider an erroneous equation which was introduced and is employed in the thesis. The first equation for the conservation of momentum, which is the incorrect Equation 6 in this report, is an excellent example. This equation is, however, in the conservative form. The corrected form of this equation is given in Equation 7 of this report, but this equation is not in the "conservative" form. The explanation of the term, "conservative", is apparent when difference equations are formed from these two differential equations. When considering Equation 6, every quantity is in differential form and the difference equation can be stated in terms of specific values of the variables which correspond to the start and to the end of a time interval, Δt. This is in contrast to Equation 7, where it is not immediately apparent as to which instant in the time interval, Δt, to employ to evaluate the quantities w and r. If the term $\frac{w^2}{r}$, is of the same order of magnitude as the other terms in the difference equation, and error of 10 per cent in the choice of the correct instant in the time interval, Δt, could result in a 10 per cent deviation in the result for each cycle that the solution is run on the computer. When the problem is solved by repeated cycling, this error may become so large that the apparent solution is not an acceptable solution for the problem.

An obvious technique exists for overcoming the difficulty of not having a "conservative" equation, but it requires several times the amount of computer time that is required for the "conservative" form of the equations. The problem must be solved for several values of Δt and for the evaluation of w and r at different instants of time within the time interval, Δt. After enough work, it will be found that the solution can be made to converge on a single solution. This will be the correct solution; but an analytical proof, that it is the correct solution, is quite difficult.

SPECULATION ON THE VALUE OF SOLUTIONS WITH THE ERRONEOUS FLOW EQUATIONS

Since the thesis was completed before the errors in it were discovered, the question arises as to the value of the solutions which are presented in it. Do the results have significance for any actual problem? The conclusion from the available evidence is that the solutions have some value and this subject will now be discussed. The results in the thesis probably deviate considerably from the impact of a sphere onto a semi-infinite slab. In contrast, the general features of the solution, without dependence on the quantitative values, are probably correct for the impact of a cylinder of infinite length on a semi-infinite slab when the axis of the cylinder is parallel to the face of the semi-infinite slab.

The results in the thesis are probably in considerable error for the impact of a sphere on a semi-infinite slab. The greatest error probably results from the initial boundary condition which assumes that the impacting element is a thin, right-cylinder instead of a wedge. In addition to the error in the initial condition, three of the four flow equations are in error for this problem. These equations were discussed in a preceding section of this report which was titled, "Correct Flow Equations in Cylindrical and Spherical Coordinates".

In contrast to the case for the sphere, the results in the thesis do appear to approximate the solution for the impact of an infinitely long cylinder onto a semi-infinite slab. The initial boundary conditions for the solution in the thesis are the initial boundary conditions for the impact of an infinitely long cylinder, with its axis parallel to the surface of the semi-infinite slab. A disc with the shape of a thin, right-cylinder is assumed to impact on the surface of the semi-infinite slab. The flow equations for the conservation of mass and the conservation of energy are correct. There is a term omitted from each of the two equations for the conservation of momentum. The effect of the omission of these two terms will be considered by reference to the equations of flow in the form that Rae[3] presents them.

Before discussing the differences between the integral of the correct and the incorrect equations, it is important to emphasize that the following comments must be considered as speculative. For calculations on the computer, the differential equations must be converted into difference equations. The integration proceeds with repeated additions of slight variations of the difference equations, and any error in the difference equation is added two to five thousand times. As a consequence, a very small error in the difference equation may produce a very significant deviation in the quantitative values in the results. With these considerations in mind, the following arguments concern only speculations on the general form of the results. The correct form of the two differential equations for the conservation of momentum are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{r\partial\theta} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{w^2}{r} = 0 \tag{8}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{r\partial\theta} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta} + \frac{uw}{r} = 0 \tag{11}$$

These equations are the correct forms of the differential equations, and are not the forms that were employed in the thesis. In the thesis, the last term in each equation is omitted. The order of magnitude of these last terms is now considered in respect to the order of magnitude of the other terms in the equations.

An inspection of the curves in the thesis show the solutions for the direction and magnitude of the velocity. It is apparent that the velocity, w, must be zero for all values of $\theta = 90^{\circ}$. This value of θ corresponds to the vertical direction on the plots which is the axis of the impact. From considerations of symmetry, there can be no flow in the horizontal direction across this vertical axis. Although w is zero, this does not require $\frac{dw}{dt}$ to be zero. As a consequence, the corrective terms in Equations 8 and 11 become zero on the vertical axis. This <u>does not mean</u> that the solution is unchanged on this line. When the angle deviates from the vertical by a small amount, the finite value of w quickly results in an error. For Equation 8 at $\theta = 85^{\circ}$, $r = 3.0\ R$, $t = 0.40 \times 10^{-9}$ seconds and $V_o = 36$ kilometers per second

$$u \frac{\partial u}{\partial r} \rightarrow 10^{14} \qquad \frac{1}{\rho} \frac{\partial p}{\partial r} \rightarrow 10^{15}$$

$$w \frac{\partial u}{r \partial \theta} \rightarrow 10^{13} \qquad - \frac{w^2}{r} \rightarrow 10^{12}$$

It must be emphasized that the sum of these terms adds to the value of $\frac{\partial u}{\partial t}$. For the other angles that are less than 85°, the order of magnitude of the terms have the following approximate values for the same time and the same initial velocity as above.

$$u \frac{\partial u}{\partial r} \rightarrow 10^{14} \qquad \frac{1}{\rho} \frac{\partial p}{\partial r} \rightarrow 10^{15}$$

$$w \frac{\partial u}{r \partial \theta} \rightarrow 10^{13} \qquad - \frac{w^2}{r} \rightarrow 10^{13}$$

As time and the value of r increases, there is an angle in the region of

45° at which u = 0. This region requires special consideration.

The relative values for Equation 11 are a little different. For angles that are less than 30°, the values of w are usually less than those for u. When $\theta = 85^{\circ}$, r = 3.0 R, $t = 0.4 \times 10^{-9}$ seconds and $V_o = 36$ kilometers per second, the terms in the momentum equation have the following order of magnitude:

$$u \frac{\partial w}{\partial r} \rightarrow 10^{12} \qquad \frac{1}{p} \frac{\partial p}{r \partial \theta} \rightarrow 10^{14}$$

$$w \frac{\partial w}{r \partial \theta} \rightarrow 10^{12} \qquad \frac{uw}{r} \rightarrow 10^{13}$$

For smaller values of θ, there is not much variation with the angle except in the vicinity of the angle at which u = 0; i.e. when w is perpendicular to r. This latter condition only holds late in the impact. For the smaller angles, the order of magnitude of the quantities is

$$u \frac{\partial w}{\partial r} \rightarrow 10^{14} \qquad \frac{1}{p} \frac{\partial p}{r \partial \theta} \rightarrow 10^{14}$$

$$w \frac{\partial w}{r \partial \theta} \rightarrow 10^{13} \qquad \frac{uw}{r} \rightarrow 10^{13}$$

There is some tendency for the magnitude of the terms to increase as the flow approaches the surface; but the ratio appears little changed.

A few comments may assist to summarize and clarify the preceding discussion. The solutions in the thesis approximates to the correct solution for the hypervelocity impact of an aluminum cylinder onto a semi-infinite slab of aluminum with the axis of the cylinder parallel to the face of the slab. The initial boundary conditions and two of the four flow equations are correct for this solution. There is, however, one term that is omitted in the radial momentum equation and another omitted term in the tangential momentum equation. The omitted term in the radial momentum equation is of the order of one per cent of the other terms in the differential equation. It is questionable if this term has a significant effect on the general features of the flow pattern. It certainly affects the quantitative values in the flow pattern, but not by several orders of magnitude. For the tangential momentum equation, the omitted term amounts to roughly 10 per cent of the other terms. Except near the surface, the tangential momentum is less than the radial momentum. Since the tangential momentum, mw, is zero on the axis of impact, the general features of the solution are probably correct on this axis. As the flow is considered at distances that are farther away from the vertical axis, there is probably a general increase in the error.

In conclusion, the general features of the impact that are illustrated in the solutions are probably correct for the hypervelocity impact on a semi-infinite slab of an infinitely long cylinder with the axis of the cylinder parallel to the surface of the slab. There is some error in the plotted results near the surface which arises from the use of the incorrect flow equations. Although the general features of the solution are probably correct, the quantitative values are certainly in error.

Added Note: It has just come to our attention that a paper on oblique, hypervelocity impact was presented at the last Symposium on Hypervelocity Impact. In this paper, Bjork et al[7] employ the impact of a cylinder with its axis parallel to the surface of the slab, as an approximation to the impact of a sphere.

ACKNOWLEDGMENT

When the suspected error in the flow equations was confirmed in the paper by Rae[3], there was considerable consternation among those that are associated with this project. The senior author wishes to take this opportunity to express his great appreciation for their wholehearted and unselfish support in clarifying the difficulties and in reformulating the problem. The contributors to the effort are a roster of the entire group, but they deserve special mention anyway. For the more involved and tedious calculations, B. A. Hardage was most valuable and did much of the work. Assisting on some phases and leading in others, L.J. Peery was most valuable. For clarifying the extent of the errors, the team of Hardage, R. E. Bruce, Peery and Lin Wang at Stillwater were kept in almost daily contact through the senior author with B. A. Sodek at Fort Worth. Since it was Sodek's thesis, it should be emphasized that he was working day and night to clear his work; as would be expected of the responsible individual that he is. Valuable assistance was also given by H. A. Reeder and R. G. McIntyre of the Mathematics Department.

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APPENDIX A

Definition of Units of Energy

The proof of the equations in Sodek's thesis may be carried through in several ways. Probably the easiest method is to show the equivalence of the equations in Sodek's thesis with the vector equations for components that are given by Rae. The three vector equations for flow are reproduced from page 192 in Richtmyer's book.[1] The equation for conservation of matter (usually called the equation of continuity) is

$$\left(\frac{\partial}{\partial t} + \vec{u}.\nabla \right) \rho = - \rho \nabla.\vec{u} \qquad \text{(a)}$$

The equation for the conservation of momentum follows from one of Newton's laws of motion and is

$$\rho \cdot \left(\frac{\partial}{\partial t} + \vec{u}.\nabla \right) \vec{u} = - \nabla p \qquad \text{(b)}$$

The equation for the conservation of energy is

$$\rho \left(\frac{\partial}{\partial t} + \vec{u}.\nabla \right) e = - p \nabla.\vec{u} \qquad \text{(c)}$$

In his book, Richtmyer gives credit for the preceding formulation of the laws to Courant and Friedrich .[2]

During the discussion in these Appendices, it will be shown that the preceding equations are equivalent to the equations for the separate components that are given by Rae.[3, 4] The deviations between the equations by Richtmyer and by Rae from the equations in Sodek's thesis will also be shown.

In the text of this report, it is stated that Equation 1 is correct. In order to show this, it will be necessary to use the results from equations that are derived from Equation b in these appendices. This will be indicated when the equations are introduced and the form that is introduced will be proven in another appendix. Sodek's equation, as reproduced by Equation 1, states

$$\rho \frac{d\epsilon}{dt} = - \nabla \cdot (p\vec{u}) \tag{d}$$

The term on the left is the total derivative. For cylindrical coordinates and for spherical coordinates when there is no flow in the tangential direction, $r\Delta\phi$,

$$\epsilon = e + \frac{1}{2} u^2 + \frac{1}{2} w^2 \tag{e}$$

where e is the internal energy, $\vec{u}$ is the component of the total velocity, $\vec{u}$, in the direction of Δr and w is the component of the total velocity in the direction of $r\Delta\theta$. Substituting equation e into equation d and expanding

$$\rho \frac{d}{dt}\left(e + \frac{1}{2} u^2_{\Delta r} + \frac{1}{2} w^2_{r\Delta\theta} \right) = - p \nabla \cdot \vec{u} - \vec{u} \cdot \nabla p$$

or

$$\rho\frac{de}{dt} + \rho u\frac{du}{dt} + \rho w\frac{dw}{dt} = - p \nabla \cdot \vec{u} - (u_r + w_{r\Delta\theta}) \cdot \left(\frac{\partial p}{\partial r}_{\Delta r} + \frac{\partial p}{r\partial\theta}_{r\Delta\theta} \right)$$

$$= -p \nabla \cdot \vec{u} - u \frac{\partial p}{\partial r} - w \frac{\partial p}{r\partial\theta} \tag{f}$$

Consider the terms on the left hand side of Equation f and expand the total differentials term by term. By definition, the first term is

$$\rho \frac{de}{dt} = \rho \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) e$$

and the second term becomes

$$\rho\, u_{\Delta r} \frac{du}{dt}\bigg|_{\Delta r} = \rho\, u_{\Delta r} \left[\frac{\partial u}{\partial t}\bigg|_{\Delta r} + (\vec{u} \cdot \nabla) u_{\Delta r}\right]$$

$$= \rho u \left[\frac{\partial u}{\partial t} + (u_{\Delta r} + w_{r\Delta\theta}) \cdot \left[\frac{\partial u}{\partial r}\bigg|_{\Delta r} + \frac{\partial u}{r\partial\theta}\bigg|_{r\Delta\theta}\right]\right]$$

$$= \rho u \frac{\partial u}{\partial t} + \rho u^2 \frac{\partial u}{\partial r} + \rho u w \frac{\partial u}{r\partial\theta}$$

The third term becomes

$$\rho w_{r\Delta\theta} \frac{dw}{dt} = \rho w_{r\Delta\theta} \left[\frac{\partial w}{\partial t}\bigg|_{r\Delta\theta} + (\vec{u} \cdot \nabla)\, w_{r\Delta\theta}\right]$$

$$= \rho w \left[\frac{\partial w}{\partial t} + (u_r + w_{r\Delta\theta}) \cdot \left(\frac{\partial w}{\partial r}\bigg|_{\Delta r} + \frac{\partial w}{r\partial\theta}\bigg|_{r\Delta\theta}\right)\right]$$

$$= \rho w \frac{\partial w}{\partial t} + \rho w u \frac{\partial w}{\partial r} + \rho w^2 \frac{\partial w}{r\partial\theta}$$

The terms on the right in equation f must also be changed in form by substitution. The terms for change are included in the second and the third terms on the right. Their values are substituted from the u-component and the w-component for the momentum, which have not yet been derived in this report. For the present, their values are obtained from Equation 2 and Equation 3 in the paper by Rae.[3] The value of the derivative in the second term on the right is given by Rae to be

$$\frac{\partial p}{\partial r} = -\rho\frac{\partial u}{\partial t} - \rho u\frac{\partial u}{\partial r} - \rho w\frac{\partial u}{r\partial\theta} + \frac{\rho w^2}{r}$$

and the value of the term in the third term on the right is given by Rae to be

$$\frac{\partial p}{r\partial\theta} = -\rho\frac{\partial w}{\partial t} - \rho u \frac{\partial w}{\partial r} - \rho w \frac{\partial w}{r\partial\theta} - \frac{\rho u w}{r}$$

equation f is now rewritten with the expanded terms that have been collected between equation f and this paragraph.

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)e+\rho u\frac{\partial u}{\partial t}+\rho u^2\frac{\partial u}{\partial r}+\rho uw\frac{\partial u}{r\partial\theta}$$

$$+\frac{\rho uw^2}{r}+\rho w\frac{\partial w}{\partial t}+\rho uw\frac{\partial w}{\partial r}+\rho w^2\frac{\partial w}{r\partial\theta}-\frac{\rho uw^2}{r}$$

$$=-p\nabla\cdot\vec{u}+\rho u\frac{\partial u}{\partial t}+\rho u^2\frac{\partial u}{\partial r}+\rho uw\frac{\partial u}{r\partial\theta}$$

$$+\rho w\frac{\partial w}{\partial t}+\rho uw\frac{\partial w}{\partial r}+\rho w^2\frac{\partial w}{r\partial\theta}$$

After the common terms are canceled, the equation simplifies to

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)e=-p\nabla\cdot\vec{u}$$

which is equation c that was written near the start of this appendix and is the equation that was taken from Richtmyer. This vector relation is correct for cylindrical coordinates and for spherical coordinates when there is no flow in the direction, $r\Delta\phi$. The conversion of this relation to the equation in cylindrical coordinates that appears in Rae's paper and the equation that appears in Sodek's thesis is postponed until consideration of the conservation of energy.

APPENDIX B

Conservation of Mass

In this appendix, it is desired to show that the vector equation from Richtmyer, equation a in Appendix A, is the equivalent of Equation 3 in the text. After this is shown, it will be found that both are equivalent to the equation for the conservation of mass in the component form that is given by Rae.[3]

As the first step in the proof, expand Equation 3 from the text

$$\frac{\partial \rho}{\partial t} = - \rho\frac{\partial u}{\partial r} - u\frac{\partial \rho}{\partial r} - \frac{\rho u}{r} - \rho\frac{\partial w}{r\partial \theta} - w\frac{\partial \rho}{r\partial \theta} \qquad \text{(a)}$$

Collect in the form that is given by Rae[3]

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} + \rho\left[\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r}\right] = 0 \qquad \text{(b)}$$

This equation is for cylindrical coordinates and corresponds to Rae's equation for cylindrical coordinates.

This same equation may now be found by expanding the vector equation by Richtmyer that is given as Equation a in Appendix A. Equation a is

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\rho = -\rho\nabla\cdot\vec{u}$$

When this equation is expanded and the vector product is expanded into cylindrical coordinates according to the relation given on page 25 of Jones[5]

$$\frac{\partial \rho}{\partial t} + (u_{\Delta r} + w_{r\Delta\theta})\cdot\left(\frac{\partial \rho}{\partial r}_{\Delta r} + \frac{\partial \rho}{r\partial \theta}_{r\Delta\theta}\right)$$

$$= -\rho\left[\frac{1}{r}\frac{\partial(ru_{\Delta r})}{\partial r} + \frac{\partial w_{r\Delta\theta}}{r\partial\theta}\right] = -\rho\frac{\partial u}{\partial r} - \frac{\rho u}{r} - \rho\frac{\partial w}{r\partial\theta}$$

Complete the product on the left and transpose terms

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + w\frac{\partial\rho}{r\partial\theta} + \rho\left(\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial\theta} + \frac{u}{r}\right) = 0 \qquad \text{(c)}$$

This equation and Equation b in this appendix are the same.

APPENDIX C

Conservation of Momentum

The two components of the momentum equation are considered separately. In the very first part of the following series of proofs, the component of the momentum in the direction of Δr is compared with the component that is given by Rae and Kirchner[4] in the Proceedings of the Sixth Conference on Hypervelocity Impact. This is a proof that Equations 7 and 8 in the text are equivalent and is a rather trivial exercise. In the second section of the first part, the component of the momentum in the direction of $r\Delta\theta$ is compared with the component that is given by Rae and Kirchner.[4] This part is proof that Equations 10 and 11 in the text are equivalent.

As in the preceding appendices, it is desirable to show that the vector equation for the conservation of momentum from Richtmyer[1], Equation b in Appendix A, is the equivalent of the two component equations for the conservation of momentum that are given by Rae.[3] This will be shown in the second part of this appendix. In defense of Sodek, it merits consideration that he had, and he used Richtmyer's equations but he did not have the Rae equations for the conservation of momentum in component form.

As the first proof in this appendix, the proof of the equivalence of Equations 7 and 8 in the text proceeds in the following manner. Start with Equation 6 and perform the partial differentiations that are indicated

$$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial p}{\partial r} - \frac{\partial(\rho r u^2)}{r\partial r} - \frac{\partial(\rho w u)}{r\partial\theta}$$

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial\rho}{\partial t} = -\frac{\partial p}{\partial r} - \frac{\rho u^2}{r} - u^2\frac{\partial\rho}{\partial r} - 2\rho u\frac{\partial u}{\partial r} - wu\frac{\partial\rho}{r\partial\theta} - \rho w\frac{\partial u}{r\partial\theta} - \rho u\frac{\partial w}{r\partial\theta} \qquad \text{(a)}$$

In the second term on the left side of the equality, substitute for $\frac{\partial\rho}{\partial t}$ from the incorrect Equation 3 in the text. Recall that Equation 3 is incorrect for cylindrical coordinates because one term is omitted. This equation is substituted in order to find the errors in the equations that Sodek used.

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial\rho}{\partial t} = \rho\frac{\partial u}{\partial t} + u\left[-\frac{\partial(r\rho u)}{r\partial r} - \frac{\partial(\rho w)}{r\partial\theta}\right]$$

$$= \rho\frac{\partial u}{\partial t} + u\left[-\frac{\rho u}{r} - \frac{\partial(\rho u)}{\partial r} - \rho\frac{\partial w}{r\partial\theta} - w\frac{\partial\rho}{r\partial\theta}\right] \qquad \text{(b)}$$

Write the expanded value from equation b into equation a

$$\rho\frac{\partial u}{\partial t} - \frac{\rho u^2}{r} - u\rho\frac{\partial u}{\partial r} - u^2\frac{\partial\rho}{\partial r} - \rho u\frac{\partial w}{r\partial\theta} - \frac{wu}{r}\frac{\partial\rho}{\partial\theta}$$

$$= \frac{\partial p}{\partial r} - \frac{\rho u^2}{r} - u^2\frac{\partial\rho}{\partial r} - 2\rho u\frac{\partial u}{\partial r} - wu\frac{\partial\rho}{r\partial\theta} - \rho w\frac{\partial u}{r\partial\theta} - \rho u\frac{\partial w}{r\partial\theta}$$

Cancel the common terms from both sides of the equality sign and collect the remaining terms on the left side of the equality sign

$$\rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} + \rho u\frac{\partial u}{\partial r} + \rho w\frac{\partial u}{r\partial\theta} = 0$$

Divide through by ρ and obtain

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{r\partial\theta} = 0 \qquad \text{(c)}$$

By comparing equation c with Equation 8, it is apparent that one term, $-\frac{w^2}{r}$, is omitted from equation c. The form of the omitted term is most important, and this term will be considered again in these comments.

Second Component of the Momentum

The proof of the equivalence of Equations 10 and 11 proceeds in the following manner. Start with Equation 9 and perform the partial differentiations that are indicated

$$\frac{\partial(\rho w)}{\partial t} = -\frac{\partial p}{r\partial\theta} - \frac{\partial(r\rho uw)}{r\partial r} - \frac{\partial(\rho w^2)}{r\partial\theta}$$

$$\rho\frac{\partial w}{\partial t} + w\frac{\partial \rho}{\partial t} = -\frac{\partial p}{r\partial\theta} - \frac{\rho uw}{r} - uw\frac{\partial\rho}{\partial r} - \rho u\frac{\partial w}{\partial r} - \rho w\frac{\partial u}{\partial r} - 2\rho w\frac{\partial w}{r\partial\theta} - w^2\frac{\partial\rho}{r\partial\theta} \qquad \text{(a)}$$

In the second term on the left side of the equality, substitute for $\frac{\partial\rho}{\partial t}$ from the incorrect Equation 3. This incorrect equation is used for the same reason as stated under the proof for the other component.

$$\rho\frac{\partial w}{\partial t} + w\left[-\frac{\rho u}{r} - u\frac{\partial\rho}{\partial r} - \rho\frac{\partial u}{\partial r} - \rho\frac{\partial w}{r\partial\theta} - w\frac{\partial\rho}{r\partial\theta}\right]$$

$$= \rho\frac{\partial w}{\partial t} - \frac{\rho uw}{r} - uw\frac{\partial\rho}{\partial r} - \rho w\frac{\partial u}{\partial r} - \rho w\frac{\partial w}{r\partial\theta} - w^2\frac{\partial\rho}{r\partial\theta}$$

$$= -\frac{\partial p}{r\partial\theta} - \frac{\rho uw}{r} - uw\frac{\partial\rho}{\partial r} - \rho u\frac{\partial w}{\partial r} - \rho w\frac{\partial u}{\partial r} - 2\rho w\frac{\partial w}{r\partial\theta} - \frac{w^2}{r}\frac{\partial\rho}{\partial\theta} \qquad \text{(b)}$$

Cancel the common terms from both sides of the equality sign and collect the remaining terms on the left side of the equality sign

$$\rho\frac{\partial w}{\partial t} + \frac{\partial p}{r\partial\theta} + \rho u\frac{\partial w}{\partial r} + \rho w\frac{\partial w}{r\partial\theta} = 0$$

Divide through by ρ

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{r\partial\theta} = 0 \qquad \text{(c)}$$

By comparing Equation c with Equation 11, it is apparent that one term, $+\frac{uw}{r}$, is omitted from Equation c.

Derivation of Component Equations from the Vector Equation

The preceding two sections in this appendix consider equations for the two components of the vector equation for the conservation of of momentum. In cylindrical coordinates with no flow along the x-direction, there are only two components of the vector equation for the conservation of momentum. It will not be shown, but the two components for the cylindrical coordinates are also the only two coordinates that are required for spherical coordinates when there is no flow in the tangential direction, $\Gamma \sin\theta \Delta \emptyset$. in the preceding proofs, it was found that the equation in the thesis omit one term from each of the two component equations, when these equations are compared with the component equations from Rae.[3] In the following proof, the vector equation for the conservation of momentum is taken from Richtmyer[1] and the two component equations from Rae are found.

The vector equation for the conservation of momentum is given by equation b in Appendix A. This equation is written

$$\rho \left(\frac{\partial}{\partial t} + \vec{u}.\nabla \right) \vec{u} = -\nabla p \qquad \text{(a)}$$

There is a short cut in the notation that is employed in this equation and in the following proof. In the above equation, $\vec{u}$ is the velocity vector. In cylindrical coordinates with no flow along the z-axis, the components of $\vec{u}$ are $u_{\Delta r}$ in the direction of Δr and $w_{r\Delta\theta}$ is the direction of $r\Delta\theta$. After these terms are introduced with the above subscripts, the following practice is to drop the subscripts and to remember the directions of u and w.

To reduce the form in equation a, introduce the vector relation

$$(\vec{u}.\nabla)\ \vec{u} = \frac{1}{2} \nabla \vec{u}.\vec{u} - \vec{u} \times \nabla \times \vec{u} \qquad \text{(b)}$$

Substitute this form in equation a and transpose to obtain

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\rho} \nabla p + \frac{1}{2} \nabla \vec{u}.\vec{u} - \vec{u} \times \nabla \times \vec{u} = 0$$

Substitute the components of u for this vector and the relation becomes

$$\frac{\partial u_{\Delta r}}{\partial t} + \frac{\partial w_{r\Delta\theta}}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r}\bigg|_{\Delta r} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta}\bigg|_{r\Delta\theta} + \frac{1}{2}\left(\frac{\partial}{\partial r}\bigg|_{\Delta r} + \frac{\partial}{r\partial\theta}\bigg|_{r\Delta\theta}\right)\left(u^2_{\Delta r} + w^2_{r\Delta\theta}\right)$$

$$- \vec{u} \times \nabla \times \vec{u} = 0$$

Partially collecting the terms, this becomes

$$\frac{1}{\rho}\frac{2p}{2r} + \frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta} + \left(u\frac{\partial u}{\partial r} + w\frac{\partial w}{\partial r}\right)\bigg|_{\Delta r} + \left(u\frac{\partial u}{r\partial\theta} + w\frac{\partial w}{r\partial\theta}\right)\bigg|_{r\Delta\theta} - \vec{u} \times \nabla \times \vec{u} = 0 \quad \text{(c)}$$

In cylindrical coordinates, the value of $\nabla \times \vec{u}$ is given by Equation 60 on page 25 of Jone's text.[5]

$$\nabla \times \vec{u} = \left(\frac{\partial \upsilon_{\Delta z}}{r\partial\theta} - \frac{\partial w_{r\Delta\theta}}{\partial z}\right)_{\Delta r} + \left(\frac{\partial u_{\Delta r}}{\partial z} - \frac{\partial \upsilon_{\Delta z}}{\partial r}\right)_{r\Delta\theta}$$

$$+ \left(\frac{1}{r}\frac{\partial(r w_{r\Delta\theta})}{\partial r} - \frac{\partial u_{\Delta r}}{r\partial\theta}\right)_{\Delta z} \quad \text{(d)}$$

Since there is no flow along the z-axis, $\upsilon_{\Delta z}=0$. In addition, there is no change in the velocity along the z-axis so the following relations hold

$$\frac{\partial w_{r\Delta\theta}}{\partial z} = \frac{\partial u_{\Delta r}}{\partial z} = 0$$

These values are now substituted in Equation d in order to obtain the following reduced form

$$\nabla \times \vec{u} = \left(\frac{\partial w}{\partial r} + \frac{w}{r} - \frac{\partial u}{r\partial\theta}\right)\bigg|_{\Delta z}$$

To complete the evaluation of $\vec{u} \times \nabla \times \vec{u}$, insert the value of $\vec{u}$ and of $\nabla \times \vec{u}$ in a matrix

$$\vec{u} \times \quad \times \vec{u} = \begin{vmatrix} \vec{\Delta r} & \vec{r\Delta\theta} & \vec{\Delta z} \\ u & w & 0 \\ 0 & 0 & \frac{\partial w}{\partial r} + \frac{w}{r} - \frac{\partial u}{r\partial\theta} \end{vmatrix}$$

$$= \left(w\frac{\partial w}{\partial r} + \frac{w^2}{r} - w\frac{\partial u}{r\partial\theta} \right)\Bigg|_{\Delta r} - \left(u\frac{\partial w}{\partial r} + \frac{wu}{r} - u\frac{\partial u}{r\partial\theta} \right)_{r\Delta\theta} \qquad \text{(e)}$$

The value in Equation e must now be substituted into Equation c. In order to obtain the change in velocity in the Δr direction, include only those terms from Equations c and e.

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\ \frac{\partial p}{\partial r} + u\frac{\partial u}{\partial r} + w\frac{\partial w}{\partial r} - w\frac{\partial w}{\partial r} - \frac{w^2}{r} + w\frac{\partial u}{r\partial\theta} = 0$$

After canceling terms, this becomes

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} + u\frac{\partial u}{\partial r} - \frac{w^2}{r} + w\frac{\partial u}{r\partial\theta} = 0$$

which is Equation 2 in Rae's paper.[4] If the change of velocity in the $r\Delta\theta$ direction is desired, collect those terms from equations e and c. The velocity in the $r\Delta\theta$ direction is

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta} + u\frac{\partial u}{r\partial\theta} + w\frac{\partial u}{r\partial\theta} + u\frac{\partial w}{\partial r} + \frac{wu}{r} - u\frac{\partial u}{r\partial\theta} = 0$$

After canceling terms, this equation becomes

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{r\partial\theta} + w\frac{\partial u}{r\partial\theta} + u\frac{\partial w}{\partial r} + \frac{wu}{r} = 0$$

which is Equation 3 in Rae's paper.[4]

APPENDIX D

Conservation of Energy

In the following discussion, two different proofs are presented. The first proof will consider the equation in the thesis for the conservation of energy. As indicated in this report, this equation is correct for cylindrical coordinates provided the total energy, ϵ, is used instead of the internal energy, e. After this correction, the equation in the thesis is correct for both cylindrical coordinates with zero flow in the Δz direction and for spherical coordinates with zero flow in the $r \sin\theta\Delta\phi$ direction. After the preceding, straightforward, but very lengthy proof, the second part of this appendix will show that Rae's component equation may be obtained from Richtmyer's vector equation for the conservation of energy.

Expansion of Equation from Thesis to Obtain Rae's Component Equation

Replace the internal energy, e, in Equation 14 in the text with the notation for the total energy, ϵ. The preceding equation is incorrect in the thesis with the internal energy, e, instead of the total energy, ϵ. After this substitution, it will be shown that the corrected equation reduces to Rae's form. The equation that is to be proven is

$$\frac{\partial(\rho\epsilon)}{\partial t} = - \frac{\partial(rpu)}{r\partial r} - \frac{\partial(pw)}{r\partial\theta} - \frac{\partial(r\rho u\epsilon)}{r\partial r} - \frac{\partial(\rho w\epsilon)}{r\partial\theta} \qquad \text{(a)}$$

Partially expand both sides of this equation

$$\rho\frac{\partial\epsilon}{\partial t} + \epsilon\frac{\partial\rho}{\partial t} = - \frac{\partial(pu)}{\partial r} - \frac{pu}{r} - p\frac{\partial w}{r\partial\theta} - w\frac{\partial p}{r\partial\theta}$$

$$- \frac{\rho u\epsilon}{r} - \rho\frac{\partial(u\epsilon)}{\partial r} - u\epsilon\frac{\partial\rho}{\partial r} - \rho w\frac{\partial\epsilon}{r\partial\theta} - \rho\epsilon\frac{\partial w}{r\partial\theta} - w\epsilon\frac{\partial\rho}{r\partial\theta}$$

Complete the expansion

$$\rho\frac{\partial \epsilon}{\partial t} + \epsilon\frac{\partial \rho}{\partial t} = - p\frac{\partial u}{\partial r} - u\frac{\partial p}{\partial r} - \frac{\rho u}{r} - p\frac{\partial w}{r\partial \theta}$$

$$- w\frac{\partial p}{r\partial \theta} - \frac{\rho u\epsilon}{r} - \rho u\frac{\partial \epsilon}{\partial r} - \rho\epsilon\frac{\partial u}{\partial r} - u\epsilon\frac{\partial \rho}{\partial r} - \rho w\frac{\partial \epsilon}{r\partial \theta} - \rho\epsilon\frac{\partial w}{r\partial \theta} - w\epsilon\frac{\partial \rho}{r\partial \theta} \qquad \text{(b)}$$

Replace the total energy, ϵ, by its value from Equation 13 in the text. The relation is

$$\epsilon = e + \frac{1}{2}u^2 + \frac{1}{2}w^2 \qquad \text{(c)}$$

The resulting equation becomes very long but everything is straightforward

$$\rho\frac{\partial e}{\partial t} + \rho u\frac{\partial u}{\partial t} + \rho w\frac{\partial w}{\partial t} + e\frac{\partial \rho}{\partial t} + \frac{1}{2}u^2\frac{\partial \rho}{\partial t} + \frac{1}{2}w^2\frac{\partial \rho}{\partial t}$$

$$= - p\frac{\partial u}{\partial r} - u\frac{\partial p}{\partial r} - \frac{pu}{r} - p\frac{\partial w}{r\partial \theta} - w\frac{\partial p}{r\partial \theta} - \rho u\frac{\partial e}{\partial r}$$

$$- \rho u^2\frac{\partial u}{\partial r} - \rho uw\frac{\partial w}{\partial r} - \rho e\frac{\partial u}{\partial r} - \frac{1}{2}\rho u^2\frac{\partial u}{\partial r} - \frac{1}{2}\rho w^2\frac{\partial u}{\partial r}$$

$$- ue\frac{\partial \rho}{\partial r} - \frac{1}{2}u^3\frac{\partial \rho}{\partial r} - \frac{1}{2}\rho uw^2\frac{\partial \rho}{\partial r} - \frac{\rho ue}{r} - \frac{1}{2}\frac{\rho u^3}{r}$$

$$- \frac{1}{2}\frac{\rho uw^2}{r} - \rho w\frac{\partial e}{r\partial \theta} - \rho wu\frac{\partial u}{r\partial \theta} - \rho w^2\frac{\partial w}{r\partial \theta} - \rho e\frac{\partial w}{r\partial \theta}$$

$$- \frac{1}{2}\rho u^2\frac{\partial w}{r\partial \theta} - \frac{1}{2}\rho w^2\frac{\partial w}{r\partial \theta} - we\frac{\partial \rho}{r\partial \theta} - \frac{1}{2}wu^2\frac{\partial \rho}{r\partial \theta} - \frac{1}{2}w^3\frac{\partial \rho}{r\partial \theta} \qquad \text{(d)}$$

On the left hand side of the preceding equation, substitute for the value of $\frac{\partial u}{\partial t}$ and $\frac{\partial w}{\partial t}$ from Rae's form of the conservation of momentum equations. In the text of this report, Equations 8 and 11 are the required equations. With this substitution, the left side of the preceding equation becomes

$$\rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t} + \frac{1}{2}u^2\frac{\partial \rho}{\partial t} + \frac{1}{2}w^2\frac{\partial \rho}{\partial t} - \rho u\left[u\frac{\partial u}{\partial r} + w\frac{\partial u}{r\partial \theta} - \frac{w^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r}\right]$$

$$- \rho w\left[u\frac{\partial w}{\partial r} + w\frac{\partial w}{r\partial \theta} + \frac{uw}{r} + \frac{1}{\rho}\frac{\partial p}{r\partial \theta}\right] = \rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t} + \frac{1}{2}u^2\frac{\partial \rho}{\partial t} + \frac{1}{2}w^2\frac{\partial \rho}{\partial t}$$

$$- \rho u^2\frac{\partial u}{\partial r} - \rho uw\frac{\partial u}{r\partial \theta} + \frac{\rho uw^2}{r} - u\frac{\partial p}{\partial r} - \rho wu\frac{\partial w}{\partial r} - \rho w^2\frac{\partial w}{r\partial \theta} - \frac{\rho uw^2}{r} - w\frac{\partial p}{r\partial \theta} \qquad \text{(e)}$$

Remove the terms that cancel between the right hand of the equality sign in Equation d and the right hand side of Equation e, which is directly above. Divide the reduced equation by ρ and rearrange the terms

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + w\frac{\partial e}{r\partial \theta} - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} \right) = - \frac{e}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} \right)$$

$$- e\left(\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r} \right) - \frac{1}{2}\frac{u^2}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} \right) - \frac{1}{2}u^2\left(\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r} \right)$$

$$- \frac{1}{2}\frac{w^2}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} \right) - \frac{1}{2}w^2\left(\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r} \right) \tag{f}$$

The equation for the conservation of mass in cylindrical coordinates is Equation 4 in the text. This equation is

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} + \rho\left(\frac{\partial u}{\partial r} + \frac{\partial w}{r\partial \theta} + \frac{u}{r} \right) = 0$$

Compare this equation with the pairs of terms on the right hand side of the equality sign in Equation f and that equation reduces to the following

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + w\frac{\partial e}{r\partial \theta} - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\frac{\partial \rho}{r\partial \theta} \right) = 0 \tag{g}$$

This corresponds to the component equation for the conservation of energy that is given by Rae.[3]

The preceding equation was derived in cylindrical coordinates, and is the same equation that is given by Rae for spherical coordinates. This is well-known, but it is desirable to emphasize that the equation is the same in both spherical and cylindrical coordinates of the type that are considered here.

Obtaining Rae's Component Equation from Richtmyer's Vector Equation

As for all other conservation equations, it is desirable to show that the component equation from Rae's papers come directly from Richtmyer's vector equation. Richtmyer's vector equation for the conservation of energy is

$$\rho\left(\frac{\partial}{\partial t} - \vec{u}.\nabla\right) e = - p \nabla . \vec{u} \qquad \text{(a)}$$

Expand the term on the left

$$\rho\left[\frac{\partial}{\partial t} + (u_{\Delta r} + w_{r\Delta\theta}) \cdot \left(\frac{\partial}{\partial r}\bigg|_{\Delta r} + \frac{\partial}{r\partial\theta}\bigg|_{r\Delta\theta}\right)\right] e = - p \nabla . \vec{u}$$

Perform the indicated operations and the equation becomes

$$\rho\frac{\partial e}{\partial t} + \rho u\frac{\partial e}{\partial r} + \rho w\frac{\partial e}{r\partial\theta} = - p \nabla . \vec{u} \qquad \text{(b)}$$

Now expand the term on the right, recalling that the divergence is in cylindrical coordinates with no spread in the ΔZ direction. Use the definition for divergence in cylindrical coordinates that is given by Jones.[5]

$$\rho\frac{\partial e}{\partial t} + \rho u\frac{\partial e}{\partial r} + \rho w\frac{\partial e}{r\partial\theta} = - p\left[\frac{1}{r}\frac{\partial(ru_{\Delta r})}{\partial r} + \frac{\partial w_{r\Delta\theta}}{r\partial\theta}\right]$$

$$= - p\left[\frac{\partial u_{\Delta r}}{\partial r} + \frac{u_{\Delta r}}{r} + \frac{\partial w_{r\Delta\theta}}{r\partial\theta}\right] \qquad \text{(c)}$$

Refer to the equation for the conservation of mass that is given in Equation 4 of the text. The equation is

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + w\frac{\partial\rho}{r\partial\theta} + \rho\left(\frac{\partial u}{\partial r} + \frac{\partial u}{r\partial\theta} + \frac{u}{r}\right) = 0 \qquad \text{(d)}$$

The term in the brackets on the right side of equation c is the same as the term in the brackets in Equation d. Substitute for the term in the brackets in Equation c.

$$\rho u\frac{\partial e}{\partial r} + \rho\frac{\partial e}{\partial t} + \rho w\frac{\partial e}{r\partial\theta} - \frac{p}{\rho}\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + w\frac{\partial\rho}{r\partial\theta}\right) = 0$$

Divide this equation by ρ , and it becomes Rae's equation for the conservation of energy

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + w\frac{\partial e}{r\partial\theta} - \frac{p}{\rho^2}\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + w\frac{\partial\rho}{r\partial\theta}\right) = 0$$